

# HIGHER STANLEY-REISNER RINGS AND TORIC RESIDUES

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ABSTRACT. We give a purely algebraic proof of the hypersurface case of Toric Residue Mirror Conjecture recently proposed by Batyrev and Materov.

## 1. INTRODUCTION

Toric Residue Mirror Conjecture (TRMC) has been formulated by Batyrev and Materov in [BM]. It is in many ways analogous to now classical calculations of (virtual) numbers of rational curves in Calabi-Yau hypersurfaces in toric varieties. In that story the generating function of the numbers of rational curves on a Calabi-Yau hypersurface is calculated in terms of the periods of the mirror family. Extensive references can be found in [BM].

Instead of using Kontsevich's moduli spaces of stable curves on the ambient toric manifold in order to define virtual numbers of curves on the hypersurface, the paper of Batyrev and Materov uses a less sophisticated toric version of moduli spaces. The resulting generating function is then conjectured to be related to the *toric residue* of the mirror family. It is important to emphasize that while the original mirror conjecture uses GKZ hypergeometric functions of [GKZ], TRMC is formulated in terms of some rational functions of several variables. This, perhaps, is the strongest indication of the relative degree of difficulty of the two conjectures. On the other hand, neither conjecture follows from the other. Similar to the usual mirror symmetry Toric Residue Mirror Conjecture can be extended to the case of Calabi-Yau complete intersections defined by nef partitions, see [BM2].

In the present paper we give a simple algebraic proof of the hypersurface case of TRMC. We build on the work done in [BM] while at the same time try to simplify it. We do not attempt to use the most geometric version of the moduli spaces of rational curves of given cohomology class on the toric variety, but are willing to use a bigger space while adjusting the virtual fundamental class on it. This allows us to

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essentially use a single cohomology space to do all the calculations in. In fact, since we are mostly interested in cohomology classes, the moduli spaces we are working with do not have a direct geometric meaning. However we feel that they greatly simplify the exposition and allow for a more conceptual understanding of TRMC.

The paper is organized as follows. In Section 2 we introduce *higher Stanley-Reisner rings*  $A_k$  of a toric variety, which is the main tool of this paper. They are closely related to the cohomology of the moduli spaces considered in [BM] but are much easier to deal with. In Section 3 we mimic the construction of [BM] to define Morrison-Plesser classes in  $A_k$  which are analogs of virtual fundamental classes on moduli spaces of Kontsevich's stable curves. Section 4 contains an explicit combinatorial calculation of the generating function of Morrison-Plesser classes for a Hessian. This is the most delicate calculation of the entire paper. Sections 2, 3 and 4 are self-contained and can be read by anyone with just a minimum background in toric geometry. By using somewhat technical results about secondary polytopes, we prove our version of TRMC in Section 5. Finally, in Section 6 we draw the connection between our definitions and that of [BM], thus establishing the hypersurface case of TRMC.

This is not the first solution of TRMC. In fact, the first proof of it belongs to András Szenes and Michèle Vergne. Although this paper is a result of a completely independent project, the author has been informed by Victor Batyrev of the work of [SV], then still in preparation. András Szenes then assured the author that the two approaches differ sufficiently to warrant the completion of the project. I thank both of them for their interest and encouragement.

## 2. HIGHER STANLEY-REISNER RINGS

In this section we describe *higher Stanley-Reisner rings*  $A_k$  of a toric variety  $\mathbb{P}_\Sigma$  for any positive integer  $k$ . The ring  $A_0$  is isomorphic to the cohomology ring of  $\mathbb{P}_\Sigma$ , and for any  $k$  the ring  $A_k$  admits a presentation inspired by Stanley-Reisner description of  $A_0$ . This is the reason behind our terminology.

Let  $\Sigma$  be a complete simplicial fan in a lattice  $M \cong \mathbb{Z}^d$  and let  $v_i, i = 1, \dots, n$  be the minimum generators of its one-dimensional cones. Let  $\mathbb{P}_\Sigma$  be the corresponding toric variety. Its cohomology is given by the Stanley-Reisner relations:

$$H^*(\mathbb{P}_\Sigma, \mathbb{C}) \cong \mathbb{C}[D_1, \dots, D_n]/I$$

where the ideal  $I$  is generated by linear relations  $\sum_{i=1}^n (\lambda \cdot v_i) D_i$  for all  $\lambda \in N = M^*$  and monomial relations  $\prod_{i=1}^n D_i^{r_i}$  over all  $\{r_i\}$  such that

*no cone of  $\Sigma$  contains all  $v_i$  for which  $r_i > 0$ .* This is a slight reformulation of the usual description, which the reader can easily verify to be equivalent.

**Definition 2.1.** For every nonnegative  $k$  we denote by  $A_k$  the quotient of the polynomial ring  $\mathbb{C}[D_1, \dots, D_n]$  by linear relations  $\sum_{i=1}^n (\lambda \cdot v_i) D_i$ ,  $\lambda \in N$  and monomial relations  $\prod_{i=1}^n D_i^{r_i}$  over all  $\{r_i\}$  such that *no cone of  $\Sigma$  contains all  $v_i$  for which  $r_i > k$* .

We will now show that  $A_k$  is isomorphic to the cohomology of some complete toric variety of dimension  $nk + d$ , defined by a fan  $\Sigma_k$  in the lattice  $M \oplus \mathbb{Z}^{nk}$ , as follows. Let  $\{e_{i,j}, 1 \leq i \leq n, 1 \leq j \leq k\}$  be a basis in  $\mathbb{Z}^{nk}$ . For each  $i$  we introduce  $e_{i,0} = -\sum_{j=1}^k e_{i,j}$ . We then consider elements

$$v_{i,j} = v_i \oplus e_{i,j}, \quad 1 \leq i \leq n, 0 \leq j \leq k.$$

Cones of the fan  $\Sigma_k$  are generated by collections of elements  $v_{i,j}$  such that the indices  $i$  that occur  $(k+1)$  times correspond to generators  $v_i$  of a cone of  $\Sigma$ .

**Proposition 2.2.** *The above described  $\Sigma_k$  is a complete simplicial fan. The cohomology ring of the corresponding toric variety  $\mathbb{P}_{\Sigma_k}$  is isomorphic to  $A_k$ .*

*Proof.* First of all, we need to see that  $\Sigma_k$  is a fan, i.e. the intersection of two cones  $C_1$  and  $C_2$  in it is again a cone in  $\Sigma_k$ . It is sufficient to show that if  $C_i$  correspond to the subsets of indices  $I$  and  $J$  of  $\{1, \dots, n\} \times \{0, \dots, k\}$  then  $C_1 \cap C_2$  is equal to the cone  $C$  spanned by  $v_{i,j}$  for  $(i, j) \in I \cap J$ . For each  $i = 1, \dots, n$  we denote by  $I_i$ ,  $J_i$  and  $(I \cap J)_i$  the  $i$ -th components of  $I$ ,  $J$  and  $I \cap J$  respectively.

It is clear that  $C \subseteq C_1 \cap C_2$ . To show the converse suppose that  $w = v \oplus \bigoplus_{i=1}^n w_i$  is in  $C_1 \cap C_2$ . We have

$$w = \sum_{i=1}^n \left( \sum_{j \in I_i} \alpha_{i,j} v_i \right) \oplus \bigoplus_{i=1}^n \sum_{j \in I_i} \alpha_{i,j} e_{i,j} = \sum_{i=1}^n \left( \sum_{j \in J_i} \beta_{i,j} v_i \right) \oplus \bigoplus_{i=1}^n \sum_{j \in J_i} \beta_{i,j} e_{i,j}$$

where all  $\alpha$  and  $\beta$  are nonnegative. For each  $i$  we have  $\alpha_{i,j} = \beta_{i,j} + \gamma_i$  for some numbers  $\gamma_i$  independent of  $j$ . We observe that if  $\gamma_i > 0$  then  $\alpha_{i,j} > 0$  for all  $j$ , so  $|I_i| = k+1$ . Similarly, if  $\gamma_i < 0$  then  $|J_i| = k+1$ . We have

$$0 = \sum_{i=1}^n \gamma_i v_i.$$

By splitting this into the sums with positive and negative  $\gamma_i$  we get

$$\sum_{i, |I_i|=k+1, \gamma_i>0} \gamma_i v_i = \sum_{i, |J_i|=k+1, \gamma_i<0} (-\gamma_i) v_i.$$

By the definition of cones in  $\Sigma_k$ , the set of  $v_i$  with  $|I_i| = k + 1$  forms a cone in  $\Sigma$ , and similarly for  $J$ . Both sides of the above identity lie in the intersection of the corresponding cones. As a result,  $\gamma_i = 0$  unless  $|I_i| = |J_i| = k + 1$ . This implies that  $\alpha_{i,j} = \beta_{i,j}$  unless  $|I_i| = |J_i| = k + 1$ . Consequently, if  $(i, j) \notin J$  then  $|J_i| < k + 1$  so  $\alpha_{i,j} = \beta_{i,j} = 0$ . Hence only nonzero  $\alpha_{i,j}$  come from  $(i, j) \in I \cap J$ . This shows that  $\Sigma_k$  is indeed a fan.

To show that  $\Sigma$  is complete, consider any  $w = v \oplus \bigoplus_{i=1}^n w_i$ . Each  $w_i$  sits in the unique cone of the standard fan for  $\mathbb{P}^k$  so it can be written in a unique way as a nonnegative linear combination of  $k$  vectors  $e_{i,j}$ . If we subtract the corresponding linear combinations of  $v_{i,j}$  we are left with an element  $v'$  of  $M_{\mathbb{R}}$ . We can write it as a positive linear combination  $v' = \sum_{v_i \in \sigma} \gamma_i v_i$  for some cone  $\sigma \in \Sigma$ . Therefore,  $v' \oplus \mathbf{0} = \sum_{v_i \in \sigma} \sum_{j=0}^k \frac{1}{k+1} \gamma_i v_{i,j}$ . It is easy to see that the resulting linear combination for  $w$  will have positive coefficients for the set of indices  $I$  such that  $|I_i| = k + 1$  iff  $v_i \in \sigma$ . Thus  $w$  lies in a cone of  $\Sigma_k$ .

To calculate the cohomology of  $\mathbb{P}_{\Sigma_k}$  we use the Stanley-Reisner presentation of it as a quotient of a polynomial ring in  $n(k + 1)$  variables  $D_{i,j}$  by linear and polynomial relations. We have linear relations  $D_{i,j_1} = D_{i,j_2}$  for all  $i, j_1$  and  $j_2$ , which come from the linear functions on each copy of  $\mathbb{Z}^k$ .

We can map  $D_i$  to  $(k + 1)D_{i,0}$  and note that linear relations coming from  $M$  give

$$\sum_{i=1}^n (\lambda \cdot v_i) D_i = 0.$$

The description of the cones of  $\Sigma_k$  then shows that the monomial relations are exactly the ones in the definition of  $A_k$ .  $\square$

**Remark 2.3.** For any  $l > 0$  the ring  $A_k$  can be naturally mapped to  $A_{k+l}$  by multiplying by  $\prod_{i=1}^n D_i^l$ . Indeed, multiplication by  $\prod_{i=1}^n D_i^l$  maps monomial relations for  $A_k$  into monomial relations for  $A_{k+l}$  so it maps the ideal of relations for  $A_k$  into that for  $A_{k+l}$ . This map is a  $\mathbb{C}[D_1, \dots, D_n]$ -module map. In what follows we will be working in the direct limit of  $A_k$  under these maps.

By Proposition 2.2, the component of degree  $nk + d$  in the graded ring  $A_k$  is one-dimensional. Moreover we have evaluation maps  $f_{\mathbb{P}_{\Sigma_k}} : A_k \rightarrow \mathbb{C}$  coming the intersection on  $\mathbb{P}_{\Sigma_k}$ . We observe that these evaluations are not quite compatible with the maps of Remark 2.3.

**Proposition 2.4.** *For any element  $a \in A_k$  and any  $l > 0$  there holds*

$$\int_{\mathbb{P}_{\Sigma_k}} a = \frac{(k+1)^{nk}}{(k+l+1)^{n(k+l+1)}} \int_{\mathbb{P}_{\Sigma_{k+l}}} a \prod_{i=1}^n D_i^l.$$

*Proof.* Since the top degree components of  $A_k$  are one-dimensional, the above statement is true up to a multiplication by a constant. Consequently, it is enough to show that

$$(2.1) \quad \int_{\mathbb{P}_{\Sigma}} a = (k+1)^{-nk} \int_{\mathbb{P}_{\Sigma_k}} a \prod_{i=1}^n D_i^k$$

for one nonzero element  $a$  of  $A_0$ . Pick a maximum cone  $\sigma \in \Sigma$  and let  $V(\sigma)$  be the *normalized volume* of the simplex generated by  $v_i \in \sigma$ , which is defined as the absolute value of the determinant of  $v_i \in \sigma$  expanded in a basis of the lattice. Then

$$\int_{\mathbb{P}_{\Sigma}} \prod_{v_i \in \sigma} D_i = V(\sigma)^{-1}.$$

On the other hand,  $\prod_{v_i \in \sigma} D_i \prod_{i=1}^n D_i^k$  corresponds to

$$\prod_{v_i \in \sigma} (k+1) D_{i,0} \prod_{i=1}^n \prod_{j=1}^k (k+1) D_{i,j}.$$

We need to find the normalized volume of the cone  $\sigma_k$  in  $\Sigma_k$  generated by  $v_{i,0}$  for  $v_i \in \sigma$  and  $v_{i,j}$  by  $1 \leq i \leq n, 1 \leq j \leq k$  in  $M \oplus \mathbb{Z}^{nk}$ . We claim it equals  $V(\sigma)(k+1)^d$ . Indeed, replacing  $v_{i,0}$  by  $\sum_{j=0}^k v_{i,j} = (k+1)v_i \oplus \mathbf{0}$  does not change the volume. Then it is easy to calculate the volume of the resulting cone. Then we have

$$\int_{\mathbb{P}_{\Sigma_k}} \prod_{v_i \in \sigma} D_i \prod_{i=1}^n D_i^k = (k+1)^{nk+d} V(\sigma_k)^{-1} = (k+1)^{nk} V(\sigma)^{-1}.$$

This shows (2.1).  $\square$

We will adjust the top class evaluation so that it is compatible with the maps of Remark 2.3.

**Definition 2.5.** For each nonnegative  $k$  we define  $\int_{A_k} : A_k \rightarrow \mathbb{C}$  by  $\int_{A_k} = (k+1)^{-nk} \int_{\mathbb{P}_{\Sigma_k}}$ . By Proposition 2.4, we have

$$(2.2) \quad \int_{A_k} a = \int_{A_{k+l}} a \prod_{i=1}^n D_i^l$$

for all  $k, l$  and all  $a \in A_k$ .

**Definition 2.6.** We denote by  $\mathcal{A}$  the direct limit of  $A_k$  taken with respect to the maps of Remark 2.3. The direct limit of  $\int_{A_k}$  gives a map  $\int_{\mathcal{A}}: \mathcal{A} \rightarrow \mathbb{C}$ . In addition  $\mathcal{A}$  inherits a structure of  $\mathbb{C}[D_1, \dots, D_n]$ -module.

**Remark 2.7.** Arguments of this section do not depend on the fact that the same lattice  $\mathbb{Z}^k$  is used for all  $i$ . In fact, for any nonnegative integers  $k_1, \dots, k_n$  one can define the fan  $\Sigma_{(k_1, \dots, k_n)}$  in the lattice

$$M \oplus \bigoplus_i \mathbb{Z}^{k_i},$$

in terms of  $v_{i,j} = v_i \oplus e_{i,j}$  for  $1 \leq i \leq n, 0 \leq j \leq k_i$ . The cohomology ring  $A_{(k_1, \dots, k_n)}$  of the corresponding toric variety  $\mathbb{P}_{\Sigma_{(k_1, \dots, k_n)}}$  is given by the usual linear relations and the relations  $\prod_i D_i^{r_i} = 0$  if the set of  $v_i$  for which  $r_i > k_i$  does not lie in a cone of  $\Sigma$ . Products of powers of  $D_i$  define maps between these rings, which are compatible with the top class evaluation, once it is adjusted by the factor  $\prod_{i=1}^n (k_i + 1)^{k_i}$ . While introducing these rings has no effect on the limit  $\mathcal{A}$ , we will use this remark in Section 6.

The following convention will be used in the later sections.

**Definition 2.8.** We define  $D_0 = -\sum_{i=1}^n D_i$  in each ring  $A_k$ .

### 3. MORRISON-PLESSER CLASSES

In what follows it will be convenient to extend the lattice  $M$  to a lattice  $\bar{M} := M \oplus \mathbb{Z}$ . We will denote by  $\mathbf{v}_k$  the elements  $v_k \oplus 1$  of  $\bar{M}$ . We will also consider  $\mathbf{v}_0 = \mathbf{0} \oplus 1 \in \bar{M}$ . In what follows we will consider linear combinations of  $\mathbf{v}_k$ , which are encoded by elements  $\beta = (b_0, b_1, \dots, b_n) \in \mathbb{Z}^{n+1}$ . From this section on we assume that the toric variety  $\mathbb{P}_{\Sigma}$  is nef-Fano. This means that  $v_i$  lie on the boundary of the convex polytope  $\Delta = \text{conv}(\{v_i, i = 1, \dots, n\})$ .

**Definition 3.1.** Let  $\beta \in \mathbb{Z}^{n+1}$  be any lattice point which satisfies  $b_0 \leq 0$ . We define the *Morrison-Plessor class* as an element of  $\mathcal{A}$  which is the image of the element in  $A_k$

$$\Phi_{\beta} = (D_0)^{-b_0} \prod_{i=1}^n D_i^{k-b_i}.$$

for some sufficiently big  $k$ . Clearly, the result is independent of a choice of  $k$ .

The following key construction is motivated by [BM]. Let  $f = 1 + \sum_{i=1}^n a_i t^{v_i}$  be a generic formal Laurent polynomial in  $t^M$ . Notice the

change of sign in our notations as compared to that of [BM]. We will also use notation  $a_0 = 1$ .

Denote by  $K$  the cone in  $\bar{M}_{\mathbb{R}}$  spanned by  $\mathbf{v}_k$ . It can be also described as  $\{c\Delta \oplus c, c \geq 0\}$ .

**Definition 3.2.** Let  $\mathbf{p} \in \bar{M}$  be a point in  $K$ . Then we define a formal Laurent series

$$\Psi_{\mathbf{p}} := \sum_{\beta: \sum_i b_i \mathbf{v}_i = -\mathbf{p}, b_0 \leq 0} \Phi_{\beta} \prod_{i=1}^n a_i^{b_i}$$

with values in  $\mathcal{A}$ .

**Remark 3.3.** The above definition gives  $\Psi_{\mathbf{p}} = 0$  if  $\mathbf{p}$  does not lie in the lattice generated by  $\mathbf{v}_i, i = 0, \dots, n$ .

**Proposition 3.4.** *For every  $\mathbf{p} \in K$  and every  $j \in \{0, \dots, n\}$  there holds*

$$D_j \Psi_{\mathbf{p}} = \alpha_j \Psi_{\mathbf{p} + \mathbf{v}_j}.$$

*Proof.* The equality has to be understood as that of formal Laurent series in  $a_1, \dots, a_n$ .

Every solution of  $\sum_i b_i \mathbf{v}_i = -\mathbf{p}$  gives a solution  $\sum_i \hat{b}_i \mathbf{v}_i = \sum_i (b_i - \delta_i^j) \mathbf{v}_i = -\mathbf{p} - \mathbf{v}_j$  and vice versa. For  $j > 0$  the coefficients  $b_0$  in these two solutions are the same, and it is straightforward to see that the elements of  $\mathcal{A}$  for the left and right hand sides of the above equation are the same.

The situation is more complicated in the case of  $j = 0$ . The summation for  $\Psi_{\mathbf{p} + \mathbf{v}_0}$  involves solutions with  $\hat{b}_0 = 0$  which have no counterpart in the summation for  $\Psi_{\mathbf{p}}$ . We will however see that these elements are in fact zero in  $\mathcal{A}$ . We will do the calculation in  $A_k$  for all  $k$  for which a given  $\Phi_{\beta}$  makes sense.

Suppose that  $\Phi_{\hat{\beta}}$  is not zero. It is proportional to the monomial

$$\prod_{i=1}^n D_i^{k - \hat{b}_i},$$

so the definition of  $A_k$  implies that the set of  $v_i$  for which  $\hat{b}_i < 0$  lies in some cone  $\sigma \in \Sigma$ . Consider the corresponding face in  $K$ , generated by all  $\mathbf{v}_i$  for which  $\hat{b}_i < 0$ . Consider its supporting hyperplane given by an element  $\mathbf{n}$  of  $\bar{N}$ . We have  $\mathbf{n} \cdot \mathbf{v}_i \geq 0$  for all  $1 \leq i \leq n$  and  $\mathbf{n} \cdot \mathbf{v}_i = 0$  if  $\hat{b}_i < 0$ . Since we have

$$\sum_{i=1}^n \hat{b}_i \mathbf{v}_i = -\mathbf{p} - \mathbf{v}_0$$

we conclude  $\mathbf{n} \cdot (\mathbf{p} + \mathbf{v}_0) \leq 0$ . However, by assumption  $\mathbf{p} \in K$ , so  $\mathbf{p} + \mathbf{v}_0$  is in the interior of  $K$  and every supporting hyperplane is strictly positive on it.  $\square$

**Proposition 3.5.** *For every  $\mathbf{p} \in K$  and every element  $\mathbf{n} \in \bar{N} = \bar{M}^*$  there holds*

$$\sum_{i=0}^n a_i(\mathbf{n} \cdot \mathbf{v}_i) \Psi_{\mathbf{p} + \mathbf{v}_i} = 0.$$

*Proof.* By Proposition 3.4 we get

$$\sum_{i=0}^n a_i(\mathbf{n} \cdot \mathbf{v}_i) \Psi_{\mathbf{p} + \mathbf{v}_i} = \sum_{i=0}^n (\mathbf{n} \cdot \mathbf{v}_i) D_i \Psi_{\mathbf{p}}.$$

For  $\mathbf{n} = \lambda \oplus 0$  the result follows from the linear relations that hold in each  $A_k$ , hence in  $\mathcal{A}$ . For  $\mathbf{n} = \mathbf{0} \oplus 1$  the result follows from the definition of  $D_0$ . By linearity, the statement holds for all  $\mathbf{n}$ .  $\square$

We will be especially interested in  $\Psi_{\mathbf{p}}$  of top degree.

**Definition 3.6.** Consider points  $\mathbf{p} \in K$  given by  $p \oplus d \in \bar{M} \cong M \oplus \mathbb{Z}$ . For each such  $\mathbf{p}$  we define a formal Laurent series in  $a_i$  by

$$\int_{\mathcal{A}} \Psi_p := \sum_{\beta: \sum_i b_i \mathbf{v}_i = -\mathbf{p}, b_0 \leq 0} \int_{\mathcal{A}} \Phi_{\beta} \prod_{i=1}^n a_i^{b_i}$$

where  $\int_{A_k}$  is the top class evaluation in  $\mathcal{A}$ , see Definition 2.6.

#### 4. HESSIANS

The goal of this section is to calculate the linear combinations of series  $\int_{\mathcal{A}} \Psi_{\mathbf{p}}$  for linear combinations of points  $\mathbf{p} \in K$  that come from certain Hessians. We will now assume that we have a reflexive polytope  $\Delta \subset M$  and let  $\mathcal{T}$  be a triangulation of  $\Delta$  whose maximum simplices contain  $\mathbf{0}$ . The vertices of  $\mathcal{T}$  are  $\{\mathbf{0}\} \cup \{v_i, i = 1, \dots, n\}$ . Points  $v_i$  include all vertices of  $\Delta$  and perhaps some other points  $v_i \in \partial\Delta$ . The triangulation  $\mathcal{T}$  induces a complete simplicial fan  $\Sigma$  in  $M_{\mathbb{R}}$ . As before, we introduce  $\bar{M} \cong M \oplus \mathbb{Z}$ ,  $\mathbf{v}_i$  and  $K$ . The reflexivity assumption on  $\Delta$  implies that every lattice point in the interior of  $K$  lies in  $\mathbf{v}_0 + K$ .

As before we fix a generic polynomial  $f = a_0 + \sum_{i=1}^n a_i v_i$  with  $a_0 = 1$ . We recall the definitions of Hessians  $H_f$  and  $H'_f$  associated to these data, see [BM]. We will give a mostly self-contained exposition, both for the benefit of the reader and to facilitate further arguments.

**Definition 4.1.** Fix a basis  $\{\mathbf{n}_1, \dots, \mathbf{n}_{d+1}\}$  of the lattice  $\bar{N} = \bar{M}^*$ . The Hessian  $H_f$  is the determinant of the square matrix of size  $d+1$  whose  $(i, j)$ -th entry is

$$\sum_{l=0}^n (\mathbf{n}_i \cdot \mathbf{v}_l)(\mathbf{n}_j \cdot \mathbf{v}_l) a_l t^{\mathbf{v}_l}$$

where  $t$  is a dummy variable.

**Remark 4.2.** The Hessian  $H_f$  can be thought of as an element of  $\mathbb{C}[K]$ . It is easy to see that  $H_f$  does not depend on the choice of the basis of  $\bar{N}$ . Indeed, any linear change in  $\mathbf{n}_i$  by a matrix  $R$  amounts to the linear change on rows and columns of the above matrix and gives an additional factor of  $(\det R)^2$  to the Hessian.

**Proposition 4.3.** [CDS] *There holds*

$$H_f = \sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \left( \prod_{i \in J} a_i \right) t^{\sum_{i \in J} \mathbf{v}_i}$$

where  $V(J)$  is the normalized  $(d+1)$ -dimensional volume of the simplex spanned by the vectors  $\mathbf{v}_i$ , for  $i \in J$ .

*Proof.* It is clear that all terms of the determinant involve at most  $(d+1)$  different  $a_i t^{\mathbf{v}_i}$ . Consequently, it is enough to find out what happens when all  $a_i$  are zero except for  $i \in J, |J| = d+1$ .

If the elements  $\mathbf{v}_i, i \in J$  are linearly dependent, there is an element of  $\mathbf{n}_1 \in \bar{N}$  that vanishes on all of them. By completing it to the basis we see that the Hessian is the determinant of a matrix with a zero first row (and zero first column). Consequently, these collections  $J$  do not contribute to the Hessian.

If the elements  $\mathbf{v}_i, i \in J$  are linearly independent, consider the dual basis of  $\bar{N}_{\mathbb{Q}}$ . The matrix will then be simply the diagonal matrix with  $(i, i)$ -th entry  $a_i t^{\mathbf{v}_i}$ . The dual basis will typically not be a basis of  $\bar{N}$ . However, a basis of  $\bar{N}$  is obtained by a linear transformation of determinant  $V(J)$  from the dual basis to  $\mathbf{v}_i, i \in J$ . The argument of Remark 4.2 then completes the proof.  $\square$

As a consequence of the above proposition,  $H_f$  is supported in the interior of  $K$ . Indeed, the terms from the boundary correspond to  $J$  with  $V(J) = 0$ . Since  $\Delta$  is reflexive,  $H_f$  is divisible by  $t^{\mathbf{v}_0}$  in  $\mathbb{C}[K]$  which allows us to introduce  $H'_f$ .

**Definition 4.4.**

$$H'_f := H_f / t^{\mathbf{v}_0} = \sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \left( \prod_{i \in J} a_i \right) t^{\sum_{i \in J} \mathbf{v}_i - \mathbf{v}_0}$$

The main result of this section is the following calculation which describes the value of  $\int_{\mathcal{A}} \Psi$  on the Hessian. We denote by  $\text{Vol}(\Delta)$  the normalized volume of  $\Delta$ .

**Theorem 4.5.**

$$\sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \left( \prod_{i \in J} a_i \right) \int_{\mathcal{A}} \Psi_{\sum_{i \in J} \mathbf{v}_i - \mathbf{v}_0} = \text{Vol}(\Delta).$$

*Proof.* The definition of  $\Psi_{\sum_{j \in J} \mathbf{v}_j - \mathbf{v}_0}$  involves the summation over  $\hat{\beta} = (\hat{b}_0, \dots, \hat{b}_n)$  with  $\hat{b}_0 \leq 0$  and

$$\sum_{i=0}^n \hat{b}_i \mathbf{v}_i = - \sum_{j \in J} \mathbf{v}_j + \mathbf{v}_0.$$

We introduce  $b_i = \hat{b}_i + \chi(i \in J) - \delta_i^0$ . Here  $\chi(i \in J)$  is 1 if  $i \in J$  and is zero otherwise and  $\delta$  is the Kronecker symbol. Then we have the sum over  $\beta = (b_0, \dots, b_n)$  with

$$\sum_{i=0}^n b_i \mathbf{v}_i = \mathbf{0}$$

and the additional assumption  $b_0 - \chi(0 \in J) + 1 \leq 0$ . This means that the sum takes place over all  $\beta$  with  $b_0 \leq 0$ , but for  $b_0 = 0$  one only uses  $J$  that contain 0.

We will analyze the contribution of  $\beta$  from the following three cases:  $b_0 < 0$ ;  $b_0 = 0, \beta \neq 0$ ;  $\beta = 0$ . We will establish the claim of the Proposition by showing that the only nonzero contribution comes from  $\beta = 0$  and equals  $\text{Vol}(\Delta)$ .

*Case  $b_0 < 0$ .* The contribution is given by

$$\begin{aligned} & \int_{A_k} \sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \prod_{i \in J} a_i \prod_{i \in J} D_0^{-\hat{b}_0} \prod_{i=1}^n (D_i^{k-\hat{b}_i} a_i^{\hat{b}_i}) \\ &= \int_{A_k} D_0^{-b_0-1} \prod_{i=1}^n (D_i^{k-b_i} a_i^{b_i}) \sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \prod_{i \in J} D_i. \end{aligned}$$

The proof of Proposition 4.3 shows that  $\sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \prod_{i \in J} D_i$  is the determinant of square matrix of size  $(d+1)$  whose  $(i, j)$ -th entry is

$$\sum_{l=0}^n (\mathbf{n}_i \cdot \mathbf{v}_l)(\mathbf{n}_j \cdot \mathbf{v}_l) D_l.$$

Here  $\{\mathbf{n}_i\}$  is an arbitrary basis of  $\bar{N}$  so we can pick  $\mathbf{n}_1$  to have  $\mathbf{n} \cdot \mathbf{v}_l = 1$  for all  $l$ . Then the first row of the matrix consists of elements that are

zero in  $A_k$ , so the determinant is zero. As a consequence, elements with  $b_0 < 0$  do not contribute to the overall sum.

*Case  $b_0 = 0, \beta \neq 0$ .* This is the most difficult part of the calculation. We again would like to show that the contribution is zero. We recall that we have a summation over the subsets  $J$  that contain 0. We abuse notations and use the same letter for the corresponding subset of  $\{1, \dots, n\}$ . We need to show that

$$(4.1) \quad \int_{A_k} \prod_{i=1}^n D_i^{k-b_i} \sum_{J \subseteq \{1, \dots, n\}, |J|=d} V(J)^2 \prod_{i \in J} D_i = 0$$

where  $V(J)$  now denotes the normalized volume of the  $d$  vectors  $v_i, i \in J$  in the lattice  $M$ .

We immediately observe that (4.1) holds unless the set of  $v_i$  such that  $b_i < 0$  lies in a cone in  $\Sigma$ . Indeed, otherwise we have  $\prod_{i=1}^n D_i^{k-b_i} = 0$  in  $A_k$ . We will denote the cone spanned by  $v_i$  with  $b_i < 0$  by  $\sigma$ . We denote by  $\theta$  the minimum face of the reflexive polytope  $\Delta$  that contains all  $v_i$  with  $b_i < 0$ . Since we have

$$\sum_{i=1}^n b_i \mathbf{v}_i = \mathbf{0},$$

all nonzero  $b_i$  correspond to elements  $v_i \in \theta$ . Indeed, there is an element  $\mathbf{n}$  in  $\bar{N}$  which vanishes on  $\mathbf{v}_i$  for  $v_i \in \theta$  and is positive on all other  $\mathbf{v}_i$ . When applied to both sides of the above equation we see that  $\sum_{v_i \notin \theta} b_i (\mathbf{n} \cdot \mathbf{v}_i) = 0$ . This shows that all terms  $b_i (\mathbf{n} \cdot \mathbf{v}_i)$  are zero, since all terms are nonnegative.

The second observation is that  $\prod_{i=1}^n D_i^{k-b_i} \prod_{i \in J} D_i$  is zero in  $A_k$  unless all  $v_i$  for  $i \in J$  lie in a codimension one face  $\theta_1 \subset \Delta$  that contains  $\theta$ . Indeed, the set of exponents that are bigger than  $k$  contains all  $v_i \in \sigma$  and all  $v_i \notin \theta, i \in J$ . To be nonzero in  $A_k$  implies that all these elements lie in a cone of  $\Sigma$ . Hence the minimum face  $\theta_2$  in  $\Delta$  that contains these elements has codimension at least one. Since this minimum face contains  $\theta$ , in fact all elements  $v_i, i \in J$  lie in  $\theta_2$ . Then any codimension one face  $\theta_1 \supseteq \theta_2$  works. In fact, for nonzero  $V(J)$  the face  $\theta_1$  is uniquely determined. As a result, we can split the summation over all  $J$  into sub-summations over  $\theta_1$ . Then (4.1) would follow from

$$(4.2) \quad \int_{A_k} \prod_{i=1}^n D_i^{k-b_i} \sum_{J \subseteq \text{vert}(\theta_1), |J|=d} V(J)^2 \prod_{i \in J} D_i = 0$$

where  $\text{vert}(\theta_1)$  is the set of indices  $i$  for which  $v_i \in \theta_1$ .

For any basis  $\{\lambda_1, \dots, \lambda_d\}$  of  $N$  consider a square matrix  $B$  with entries

$$B_{ij} = \sum_{v_l \in \theta_1} (\lambda_i \cdot v_l)(\lambda_j \cdot v_l) D_l.$$

Similarly to the proof of Proposition 4.3, we can see that

$$\sum_{J \subseteq \text{vert}(\theta_1), |J|=d} V(J)^2 \prod_{i \in J} D_i = \text{Det}(B).$$

Since we are only trying to show that this determinant is zero, we could use a basis of  $N_{\mathbb{Q}}$  instead of  $N$ . We will pick a special basis as follows. Element  $\lambda_1$  will be equal to 1 on all  $v_i \in \theta_1$ . Elements  $\lambda_2, \dots, \lambda_r$  where  $r = \dim(\theta_1) - \dim(\theta) + 1$  will be zero on  $\theta \subseteq \theta_1$ . It is easy to see that these elements could be completed to a basis, if  $\theta$  is non-empty which is guaranteed by  $\beta \neq 0$ .

The first row of the matrix  $B$  consists of  $B_{1,j} = \sum_{v_l \in \theta_1} (\lambda_j \cdot v_l) D_l$ . These elements equal  $-\sum_{v_l \notin \theta_1} (\lambda_j \cdot v_l) D_l$  due to linear relations in  $A_k$ . We are going to replace the first row of  $B$  by the above elements of  $\mathbb{C}[D]$  and call the resulting matrix  $B'$ . We then calculate the determinant of  $B'$  as an element in  $\mathbb{C}[D]$ . We claim that all monomials in  $D_i$  that appears in the resulting expression do not have  $v_i$  lie in *any* face of  $\Delta$  that contains  $\theta$ . Consequently, the above arguments show that their contribution to the left hand side of (4.2) are zero.

To substantiate our claim, we expand the determinant of  $B'$  along the first  $r$  rows. It is sufficient to show that all the  $r \times r$  minors of the first  $r$  rows of  $B'$  have nonzero coefficients only by monomials

$$\prod_{i=1}^r D_{l_i}$$

such that no proper face  $\theta_2 \supseteq \theta$  contains all of  $v_{l_i}$ . Suppose such monomial and such  $\theta_2$  exist. As in the proof of Proposition 4.3 we can replace the rows  $2, \dots, r$  of  $B'$  by keeping only the linear combinations of  $D_{l_i}$ . We call the resulting matrix  $B''$ . The face  $\theta_2$  can not equal  $\theta_1$ , since the first row of  $B'$  has  $D_l$  with  $v_l \notin \theta_1$ . The intersection of  $\theta_1$  and  $\theta_2$  is a proper subface of  $\theta_1$ . Consequently, there is a linear combination of  $\lambda_2, \dots, \lambda_r$  which vanishes on  $\theta_1 \cap \theta_2$ . By taking the appropriate linear combination of the rows  $2, \dots, r$  of  $B''$ , we get a zero row, which means that the monomial  $\prod_{i=1}^r D_{l_i}$  occurs with zero coefficient.

*Case  $\beta = 0$ .* This case is essentially covered in [BM] but we reproduce the argument here. The contribution equals

$$\begin{aligned} \int_{A_k} \prod_{i=1}^n D_i^k \sum_{J \subseteq \{1, \dots, n\}, |J|=d} V(J)^2 \prod_{i \in J} D_i &= \int_{\mathbb{P}^\Sigma} \prod_{i=1}^n \sum_{J \subseteq \{1, \dots, n\}, |J|=d} V(J)^2 \prod_{i \in J} D_i \\ &= \sum_{\sigma \in \Sigma, \dim \sigma = d} V(\sigma)^2 \int_{\mathbb{P}^\Sigma} \prod_{v_i \in \sigma} D_i = \sum_{\sigma \in \Sigma, \dim \sigma = d} V(\sigma) = \text{Vol}(\Delta). \end{aligned}$$

Here  $V(\sigma)$  is the normalized volume of the corresponding simplex of the triangulation. This finishes the proof of Theorem 4.5.  $\square$

## 5. TORIC RESIDUE MIRROR CONJECTURE

As before, we are working with a reflexive polytope  $\Delta$ , a subset  $\{v_i\}$  of its boundary points and a triangulation  $\mathcal{T}$  of  $\Delta$  whose maximum simplices contain  $\mathbf{0}$ . We will combine together the results of Sections 3 and 4 to establish Theorem 5.2 which is the main result of this paper. We refer to this theorem as Toric Residue Mirror Conjecture. We will explain in Section 6 that it implies the original conjecture of [BM].

To explain the statement of Theorem 5.2 we need to introduce the notion of *toric residues*, as described in [BM]. The cone  $K$  in the lattice  $\bar{M} \cong M \oplus \mathbb{Z}$  is defined as the span of  $\Delta \oplus 1$ . We introduce  $\mathbf{v}_i = v_i \oplus 1$  and  $\mathbf{v}_0 = \mathbf{0} \oplus 1$ . As before, we consider a generic Laurent polynomial  $f = 1 + \sum_{i=1}^n a_i t^{v_i}$  and set  $a_0 = 1$ . Pick a basis  $\mathbf{n}_0, \dots, \mathbf{n}_d$  of  $\bar{N} = \bar{M}^*$ . The quotient of the graded ring  $\mathbb{C}[K]$  by the elements

$$Z_j = \sum_{i=0}^n (\mathbf{n}_j \cdot \mathbf{v}_i) a_i t^{\mathbf{v}_i}, \quad i = 0, \dots, d$$

is a graded Gorenstein Artin ring, and its degree  $d$  component is one-dimensional. It is spanned by the Hessian  $H'_f$  considered in Section 4. *Toric residue* is a map

$$\text{Res}_f : \mathbb{C}[K]_d \rightarrow \mathbb{C}$$

uniquely defined by its vanishing on the degree  $d$  component of the ideal  $\langle Z_0, \dots, Z_d \rangle \mathbb{C}[K]$  and by the normalization

$$\text{Res}_f(H'_f) = \text{Vol}(\Delta).$$

For a given point  $\mathbf{p} = p \oplus d$  in  $K$  the value of  $\text{Res}_f(t^\mathbf{p})$  is a rational function in  $a_i$  with denominator equal to the *principal determinant*  $E = E(a_1, \dots, a_n)$ , see [BM, Theorem 2.9].

The principal determinant  $E$  is a Laurent polynomial in  $a_i$ . We can think of its monomials as being indexed by a lattice  $\mathbb{Z}^n$  with basis  $\{e_i\}$ .

Vertices of the Newton polytope of  $E$  (also called *secondary polytope*) are exactly *characteristic functions*

$$\chi_{\mathcal{T}} = \sum_i \sum_{\sigma: v_i \in \sigma \in \mathcal{T}, \dim \sigma = d} V(\sigma) e_i$$

which correspond to *regular* triangulations  $\mathcal{T}$  of  $\Delta$  whose set of vertices is a subset of  $\{\mathbf{0}\} \cup \{v_i, i = 1, \dots, n\}$ . The proofs of these statements are contained in [GKZ1], see also [BM]. Here the triangulation is called regular if there exists a convex piece-wise linear function  $\Delta \rightarrow \mathbb{R}$  whose domains of linearity are precisely the simplices of  $\mathcal{T}$ .

From now on we will assume that the triangulation  $\mathcal{T}$  is regular, which also means that the toric variety  $\mathbb{P}_{\Sigma}$  is projective. The rational function  $\text{Res}_f(t^{\mathbf{p}})$  can be expanded in a Laurent series expansion in the normal cone of the vertex  $\chi_{\mathcal{T}}$  of the Newton polytope of  $E$ , see [BM, Definition 4.5]. This cone can be described as follows. Let  $h : M \rightarrow \mathbb{Z}$  be a convex function which is linear on the cones of  $\Sigma$  and corresponds to an ample divisor on  $\mathbb{P}_{\Sigma}$ . We extend  $h$  to  $M_{\mathbb{R}}$  by linearity. Then for any set of points  $y_i \in M_{\mathbb{R}}$  and any positive numbers  $\alpha_i$  there holds

$$(5.1) \quad h\left(\sum_i \alpha_i y_i\right) \leq \sum_i \alpha_i h(y_i)$$

with the equality achieved if and only if there is a cone  $\sigma \in \Sigma$  that contains all points  $y_i$ . The set of such convex functions will be called the ample cone of  $\mathcal{T}$  and will be denoted by  $C_{\mathcal{T}}^{ample}$ .

The following proposition is well-known, but we were unable to find a good reference in the literature.

**Proposition 5.1.** (*see also* [BM, Remark 4.7]) *The normal cone  $C_{\mathcal{T}}$  to the vertex  $\chi_{\mathcal{T}}$  in  $\mathbb{Z}^n$  can be characterized by the condition*

$$\sum_{i=1}^n \alpha_i e_i \in C_{\mathcal{T}} \Leftrightarrow \sum_{i=1}^n \alpha_i v_i = \mathbf{0} \text{ and } \sum_{i=1}^n \alpha_i h(v_i) \geq 0 \text{ for all } h \in C_{\mathcal{T}}^{ample}.$$

*Proof.* The normal cone is generated by the differences  $\chi_{\mathcal{T}_1} - \chi_{\mathcal{T}}$  of the characteristic functions over all regular triangulations  $\mathcal{T}_1$  with the same set of vertices. For every  $\mathcal{T}$  if  $\chi_{\mathcal{T}} = \sum_{i=1}^n \alpha_i e_i$ , then

$$\sum_{i=1}^n \alpha_i v_i = \sum_{\sigma \in \mathcal{T}, \dim \sigma = d} V(\sigma) \sum_{v_i \in \sigma} v_i$$

is up to a constant the baricenter of  $\Delta$  and is therefore independent of the triangulation. Consequently, all differences between various  $\chi_{\mathcal{T}}$  satisfy  $\sum_{i=1}^n \alpha_i v_i = \mathbf{0}$ .

For every triangulation  $\mathcal{T}$  and every collection of values  $(h_0, \dots, h_n) \in \mathbb{R}^{n+1}$  there is a unique piecewise linear function  $h_{\mathcal{T}}$  on  $\Delta$  which takes values  $h_i$  on the vertices  $v_i$  and  $v_0 = \mathbf{0}$  of  $\mathcal{T}$  and is linear on simplices of  $\mathcal{T}$ . Moreover, for a general collection  $(h_i)$  this function  $h_{\mathcal{T}}$  is convex for exactly one configuration which corresponds to the "bottom" of the convex hull of  $\{v_i \oplus h_i\} \in M_{\mathbb{R}} \oplus \mathbb{R}$ . For such  $\mathcal{T}$  and  $h_{\mathcal{T}}$  the value  $h_{\mathcal{T}}(p)$  is the smallest among all possible values of  $h_{\mathcal{T}^1}$  for all triangulations  $\mathcal{T}^1$ . The value of  $\sum_i \alpha_i h(v_i)$  for a characteristic function of  $\mathcal{T}$  is easily seen to equal the integral of  $h_{\mathcal{T}}$  over  $\Delta$ . Consequently, it is the smallest of these values among all vertices of the Newton polytope of  $E$  if and only if  $h$  is convex on  $\mathcal{T}$ , and vice versa.  $\square$

We are now ready to state our main result.

**Theorem 5.2.** *Let  $\mathcal{T}$  be a regular triangulation of a reflexive polytope  $\Delta$ . For every  $\mathbf{p} = p \oplus d \in K$  which lies in the lattice spanned by  $\mathbf{v}_0, \dots, \mathbf{v}_n$  the formal Laurent series  $\int_{\mathcal{A}} \Psi_{\mathbf{p}}$  is the expansion of the rational function  $\text{Res}_f(t^{\mathbf{p}})$  at the vertex  $\chi_{\mathcal{T}}$  of the Newton polytope of  $E$ .*

**Remark 5.3.** Theorem 5.2 implies that the series  $\int_{\mathcal{A}} \Psi_{\mathbf{p}}$  is in fact convergent in some open set of  $(a_1, \dots, a_n) \in \mathbb{C}^n$ .

*Proof.* First of all, we can restrict our attention to the sublattice of  $\bar{M}$  spanned by  $\mathbf{v}_i$ . All statements about the residues remain unchanged, except for a possible change in the normalization by the index of the sublattice. This is quite different from only looking at the subring of  $\mathbb{C}[K]$  generated by  $t^{\mathbf{v}_i}$ , the latter may fail to have a one-dimensional degree  $d$  component of the quotient by the ideal  $\langle Z_1, \dots, Z_d \rangle$ .

Let  $\mathbf{p} = p \oplus d$  be a point in  $K$ .

**Lemma 5.4.** *Let  $\beta \in \mathbb{Z}^{n+1}$  satisfy  $\sum_{i=0}^n b_i \mathbf{v}_i = -\mathbf{p}$  and  $b_0 \leq 0$ . Then  $\int_{\mathcal{A}} \Phi_{\beta}$  is zero unless*

$$\sum_{i=1}^n b_i h(v_i) + h(p) \geq 0.$$

for any  $h \in C_{\mathcal{T}}^{\text{ample}}$ .

*Proof.* We have  $\sum_{i=1}^n b_i v_i + p = 0$ . We can rewrite it as

$$\sum_{i, b_i \geq 0} b_i v_i + p = \sum_{i, b_i < 0} (-b_i) v_i.$$

If  $\int_{\mathcal{A}} \Phi_\beta \neq 0$ , then all  $v_i$  for which  $b_i < 0$  lie in a cone of  $\Sigma$ . We then use (5.1) to show that

$$\begin{aligned} \sum_{i=1}^n b_i h(v_i) + h(p) &= \left( \sum_{i, b_i \geq 0} b_i h(v_i) + h(p) \right) - \left( \sum_{i, b_i < 0} (-b_i) h(v_i) \right) \\ &= \left( \sum_{i, b_i \geq 0} b_i h(v_i) + h(p) \right) - h \left( \sum_{i, b_i \geq 0} b_i v_i + p \right) \geq 0. \end{aligned}$$

□

*Proof of Theorem 5.2 continues.* The above lemma implies that the formal Laurent series  $\int_{\mathcal{A}} \Psi_{\mathbf{p}}$  are in fact supported in a finite number of affine shifts of the cone  $C_T$ . The same is true for the Laurent expansions of  $\text{Res}_f(t^{\mathbf{p}})$ . We denote by  $F_{\mathbf{p}} = \int_{\mathcal{A}} \Psi_{\mathbf{p}} - \text{Res}_f(t^{\mathbf{p}})$  the differences and observe that Proposition 3.5, Theorem 4.5 and the definition of the toric residue imply that

- For all  $\mathbf{p}_1 = p_1 \oplus (d-1)$  there holds

$$\sum_{i=0}^n a_i (\mathbf{n} \cdot \mathbf{v}_i) F_{\mathbf{p}_1 + \mathbf{v}_i} = 0.$$

- 

$$\sum_{J \subseteq \{0, \dots, n\}, |J|=d+1} V(J)^2 \left( \prod_{i \in J} a_i \right) F_{\sum_{i \in J} \mathbf{v}_i - \mathbf{v}_0} = 0.$$

Since for generic  $\{a_i\}$  the elements  $\sum_{i=0}^n a_i (\mathbf{n} \cdot \mathbf{v}_i) t^{\mathbf{p}_1 + \mathbf{v}_i}$  and  $H'_f$  generate  $\mathbb{C}[K]_{\deg=d}$ , the element  $t^{\mathbf{p}}$  can be written as their linear combination with coefficients being rational functions in  $\{a_i\}$ . Consequently, there is a polynomial  $G(a_1, \dots, a_n)$  such that  $G(a_1, \dots, a_n) F_{\mathbf{p}} = 0$ . We remark that a multiplication of a formal Laurent series by a polynomial is well-defined. We now use the fact that  $F$  is supported in a finite number of affine shifts of  $C_T$ . Let  $\phi : \mathbb{Z}^n \rightarrow \mathbb{R}$  be a generic linear function which is positive on  $C_T - \{\mathbf{0}\}$ . If  $F_{\mathbf{p}} \neq 0$  then there is a term  $c_\alpha a^\alpha$  of  $F_{\mathbf{p}}$  which has the smallest value of  $h(\alpha)$  among the terms with  $c_\alpha \neq 0$ . The same can be said about  $G$ , and it is easy to see that the product of these terms in  $F_{\mathbf{p}} G$  does not cancel.

We have thus shown that  $F_{\mathbf{p}} = 0$  for all  $\mathbf{p} = p \oplus d$ , which proves the theorem. □

**Corollary 5.5.** *Let  $P(x_1, \dots, x_n) \in \mathbb{Q}[x_1, \dots, x_n]$  be a degree  $d$  polynomial. Then the Laurent expansion of the toric residue  $R_P(a) = \text{Res}_f P(a_1 t^{\mathbf{v}_1}, \dots, a_n t^{\mathbf{v}_n})$  at the vertex  $\chi_T$  is equal to*

$$\sum_{\beta: \sum_i b_i \mathbf{v}_i = 0, b_0 \leq 0} \int_{A_k} P(D_1, \dots, D_n) D_0^{-b_0} \prod_{i=1}^n D_i^{k-b_i} \prod_{i=1}^n a_i^{b_i}$$

where  $k$  is to be taken sufficiently big for each  $\beta$ .

*Proof.* The statement follows from Proposition 3.4, Theorem 5.2 and definitions of  $\int_{\mathcal{A}} \Psi$ .  $\square$

## 6. CONNECTION TO THE ORIGINAL VERSION OF TRMC

In this section we will establish the connection between Theorem 5.2 and Toric Mirror Symmetry Conjecture of [BM]. We will also remark on the complete intersection case.

We are working in the notations of the previous section.

**Proposition 6.1.** *Assume that  $v_i$  generate  $M$ . Then Corollary 5.5 implies Conjecture 4.6 of [BM].*

*Proof.* First we observe that  $\sum_{i=0}^n b_i v_i = \mathbf{0} \in \bar{M}$  is equivalent to  $\sum_{i=1}^n b_i v_i = \mathbf{0} \in M$  and  $-b_0 = b_1 + \dots + b_n$ . The argument of Lemma 5.4 shows that the summation in Corollary 5.5 can be taken over the effective classes  $\beta$ , see [BM]. We also observe that the change in sign of  $a_i$  in our notations accounts for the factor  $(-1)^d$  of [BM, Conjecture 4.6] and introduces an extra factor  $(-1)^{b_1+\dots+b_n} = (-1)^{-b_0}$  to each term.

As a result, it remains to connect

$$\int_{A_k} P(D_1, \dots, D_n) (-D_0)^{-b_0} \prod_{i=1}^n D_i^{k-b_i}$$

with

$$\int_{\mathbb{P}_{\beta}} P(D_1, \dots, D_n) (D_1 + \dots + D_n)^{b_1+\dots+b_n} \prod_{b_j < 0} D_j^{-b_j-1}$$

where  $\mathbb{P}_{\beta}$  is defined in [BM, Proposition 3.2 and Definition 3.3].

Without loss of generality we can assume that  $b_1, \dots, b_{n-r} \geq 0$  and  $b_{n-r+1}, \dots, b_n < 0$ . We consider the variety  $\mathbb{P}_{\Sigma_{(b_1, \dots, b_{n-r}, 0, \dots, 0)}}$  of Remark 2.7. It is given by a fan  $\Sigma_{\beta}$  in the lattice

$$M \oplus \bigoplus_{i=1}^{n-r} \mathbb{Z}^{b_i}$$

The vertices of the fan are  $v_{i,j} = v_i \oplus e_{i,j}$  for  $i \leq n-r$ ,  $b_i > 0$  and just  $v_i$  for  $i > n-r$  or  $i \leq n-r$ ,  $b_i = 0$ . The cones are given by the condition that the indices  $i$  for which all  $v_{i,j}$  are used lie in a cone of  $\Sigma$ .

We claim that *up to a finite index change of lattice*  $\mathbb{P}_{\beta}$  is isomorphic to the toric subvariety  $\mathbb{P}'_{\beta}$  in  $\mathbb{P}_{\Sigma_{(b_1, \dots, b_{n-r}, 0, \dots, 0)}}$  which corresponds to the cone generated by  $v_i$ ,  $i > n-r$ . This variety is empty if  $v_{n-r+1}, \dots, v_n$

do not form a cone in  $\Sigma$  and is otherwise given by the image of the link of  $\sigma = \text{Span}(v_{n-r+1}, \dots, v_n)$  in  $\Sigma_{(b_1, \dots, b_{n-r}, 0, \dots, 0)}$  modulo the lattice  $M_1 = (\mathbb{Q}v_{n-r+1} + \dots + \mathbb{Q}v_n) \cap M$ . The lattice of  $\mathbb{P}'_\beta$  is the quotient of the lattice  $M \oplus \bigoplus_{i=1}^{n-r} \mathbb{Z}^{b_i}$  by  $M_1$ .

We need to recall the definition of  $\mathbb{P}_\beta$  from [BM]. Consider the lattice

$$\mathbb{Z}(\beta) := \bigoplus_{i=1}^{n-r} \mathbb{Z}_i(\beta) \cong \bigoplus_{i=1}^{n-r} \mathbb{Z}^{b_i+1}$$

with the basis  $w_j^{(i)}$ . Our notations differ by a switch of  $i$  and  $j$  from that of [BM]. Consider the sublattice in  $\mathbb{Z}(\beta)$  defined by the condition

$$\sum_{i=1}^{n-r} c_i \left( \sum_{j=0}^{b_i} w_j^{(i)} \right) = 0$$

for every solution of  $\sum_{i=1}^n c_i v_i = \mathbf{0}$ . Then the fan of  $\mathbb{P}_\beta$  lives in the dual  $L$  of this sublattice, which can be thought of as the quotient of the lattice  $\mathbb{Z}(\beta)^*$  by elements

$$(6.1) \quad \sum_{i=1}^{n-r} c_i y_i$$

for  $\sum_{i=1}^n c_i v_i = \mathbf{0}$  where  $y_i = \sum_{j=0}^{b_i} w_j^{(i)*}$ . Notice that the images of the elements  $\frac{1}{b_i+1} v_{i,j}$ ,  $i \leq n-r$  in  $M_{\mathbb{Q}} \oplus \bigoplus_{i=1}^{n-r} \mathbb{Q}^{b_i}/(M_1)_{\mathbb{Q}}$  satisfy the same relations (6.1) as the images of  $w_j^{(i)}$  in  $L$ . Maximum-dimensional cones of the fan of  $\mathbb{P}_\beta$  are described in the proof of Proposition 3.2 of [BM] and it is easy to see that they are in one-to-one correspondence with the cones of the fan of  $\mathbb{P}'_\beta$ . We also remark that the projection in that proof shows that if  $v_{n-r+1}, \dots, v_n$  do not lie in a cone of  $\Sigma$ ,  $\mathbb{P}_\beta$  is empty.

Because varieties  $\mathbb{P}_\beta$  and  $\mathbb{P}'_\beta$  are isomorphic up to a lattice change, their cohomology rings are isomorphic with the isomorphism mapping  $D_{i,j}$  to  $\frac{1}{b_i+1} D_j^{(i)}$  which corresponds to the element  $w_j^{(i)*}$ . As a result, their Stanley-Reisner descriptions in terms of the polynomial ring  $\mathbb{C}[D_1, \dots, D_n]$  have exactly the same ideal, which includes  $D_{n-r+1}, \dots, D_n$ . Now we only need to make sure that the top class evaluations are the same, which again amounts to an index calculation for some maximum-dimensional cone. The details are left to the reader.  $\square$

**Remark 6.2.** In general, it appears that Conjecture 4.6 of [BM] needs to be adjusted by the index of the sublattice of  $M$  generated by  $v_i$  inside the lattice  $M$ . For instance,  $\mathbb{P}_0$  is in general not isomorphic to  $\mathbb{P}_\Sigma$  but is rather a non-ramified abelian cover of  $\mathbb{P}_\Sigma$ .

**Remark 6.3.** While higher Stanley-Reisner rings  $A_k$  are easier to define and work with, they lack the direct geometric motivation of the

toric moduli spaces  $\mathbb{P}_\beta$  of [BM]. It is also quite possible that they are better thought of as Deligne-Mumford stacks, see [BCS].

**Remark 6.4.** It is reasonable to expect that the techniques of this paper are applicable to the complete intersection case of the conjecture, see [BM2].

**Remark 6.5.** It would be interesting to try to apply higher Stanley-Reisner rings to other open problems in the area. For example one can try to use them to bound the regularity of the subring of  $\mathbb{C}[K]$  generated by  $t^{\mathbf{v}_i}$ . Surprisingly little is known about this toric case of the more general Eisenbud-Goto conjecture [EG].

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